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FACTORIZATION OF CURVATURE OPERATORS

BY

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ABSTRACT. Let V be a real finite-dimensional vector space with inner product and let R be a curvature operator, i.e., a symmetric linear map of the bivector space $\Lambda^2 V$ into itself. Necessary and sufficient conditions are given for R to admit factorization as $R = \Lambda^2 L$, with L a symmetric linear map of V into itself. This yields a new characterization of Riemannian manifolds that admit local isometric embedding as hypersurfaces of Euclidean space.

1. Introduction. A curvature operator is a symmetric linear map $R: \Lambda^2 V \to \Lambda^2 V$, where V is a real n-dimensional vector space with inner product and $\Lambda^2 V$ is the associated $\binom{n}{2}$-dimensional bivector space, supplied with the inner product induced from V. The problem solved in this paper is the following: to characterize those curvature operators $R: \Lambda^2 V \to \Lambda^2 V$ that can be factored as $R = \Lambda^2 L$ for suitable symmetric linear maps $L: V \to V$.

This problem is closely related to the local embedding problem for Riemannian manifolds. Namely, if M^n is an abstract Riemannian manifold, one asks for the least integer p for which M^n is locally isometrically embeddable into Euclidean space E^{n+p}. It is known that $p \leqslant \frac{1}{2}n(n+1)$ in the C^∞ case, and that p depends on the curvature tensor of M^n. At each point of M^n, this tensor can be represented as a symmetric linear map $R: \Lambda^2 V \to \Lambda^2 V$, where V is the tangent space, endowed with the inner product given by the metric tensor. If M^n is isometrically embeddable into E^{n+p}, then R must at each point admit factorization as a sum $R = \Lambda^2 L_1 + \cdots + \Lambda^2 L_p$ for some symmetric linear maps $L_i: V \to V$. In general, it is not known how to determine the minimum-length factorization of R. The purpose of this paper is to characterize those R which admit the shortest possible factorization, namely, $p = 1$.

For this case, the factorization is in fact equivalent to local embeddability into E^{n+1}. Namely, T. Y. Thomas has shown **[8]** that an abstract Riemannian manifold M^n with rank $R \geqslant 6$ admits local isometric embedding into E^{n+1} if and only if R can be factored as $R = \Lambda^2 L$ at each point. *Hence, the main result of the present paper provides necessary and sufficient conditions for local isometric embedding into* E^{n+1} *when curvature has rank* $\geqslant 6$.

The $p = 1$ local embedding problem has been considered before by several authors, each working with quite different methods. Namely, T. Y. Thomas **[8]** obtained a solution by algebraic means, involving certain 3×3 determinants in the

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matrix components of R. Rozenson [7] noted that the determinants in Thomas' solution were not coordinate-invariant; she obtained some rather complicated tensor expressions involving R, and gave a solution in terms of these tensors. Yanenko [**11**] gave essentially the same solution as Thomas, although he worked with differential 2-forms rather than just the matrix components of R.

The methods of this paper are different. The solution given here involves the properties of R in relation both to the vector space structure and to the exterior product structure of the space $\Lambda^2 V$. In fact, the principal condition on R is that it must preserve the decomposability of bivectors. It is hoped that the methods used here will be useful also for studying the $p \geqslant 2$ case.

The main result of the paper is Theorem 7.8, contained in §7 below.

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2. Plan of work. Let R: $\Lambda^2 V \to \Lambda^2 V$ be a curvature operator. Our problem is to find conditions on R that permit R to be factored as $R = \Lambda^2 L$, for a suitable linear map L: $V \to V$. This problem was solved in [**9**] for the case of nonsingular R, by means of a two-stage procedure. Namely, R was first factored as $R = \pm \Lambda^2 L$ and then the minus sign was removed. The results obtained in [**9**] can be applied to the case of an R with nontrivial kernel as follows.

Since R is symmetric, the vector space $\Lambda^2 V$ admits the orthogonal decomposition $\Lambda^2 V = \ker R + \operatorname{im} R$, and R splits into the sum $R = R_0 + R_1$, where R_1: $\operatorname{im} R \to \operatorname{im} R$ is a nonsingular linear map, and R_0: $\ker R \to \operatorname{im} R$ is identically zero. If it happens that the subspace $\operatorname{im} R$ of $\Lambda^2 V$ is of the form $\Lambda^2 U$ for some subspace U of V, then the first result of [**9**] (cf. Proposition 7.1 below) can be used to factor the map R_1 as $R_1 = \pm \Lambda^2 L_1$, for a suitable linear map L_1: $U \to U$. The full factorization $R = \pm \Lambda^2 L$ is then obtained by setting $L = L_0 + L_1$, where L_0: $V \to U$ is the zero map. Finally, the minus sign can be removed by the same technique as in [**9**].

Therefore, the first task in this paper is to find conditions on R that allow its image to be factored as $\operatorname{im} R = \Lambda^2 U$, for a suitable subspace U of V.

We assume the elements of multilinear algebra (cf. [**6**], [**10**], and [**4**] especially for inner products).

3. Decomposability preserving maps and the subspace U. A linear map R: $\Lambda^2 V \to \Lambda^2 V$ is said to *preserve decomposability* if each nonzero decomposable bivector either has a nonzero decomposable image or gets mapped to zero. Since we include zero in the set G of decomposable bivectors, this happens precisely when $R(G) \subset G$.

Preservation of decomposability is the main condition on R which is needed to factor the image of R as $\operatorname{im} R = \Lambda^2 U$. In this section we give some equivalent formulations and general consequences of this condition, and we also define the subspace $U \subset V$. The proofs are omitted, since they are straightforward. We remind the reader that decomposable bivectors α, β are called *adjacent* if $\alpha\beta = 0$.

PROPOSITION 3.1. (i) *If R preserves decomposability, then it also preserves adjacence.*

(ii) *R preserves decomposability if and only if* $R(x_1x_2)R(x_3x_4) = -R(x_1x_3)R(x_2x_4)$ *for all* $x_1, x_2, x_3, x_4 \in V$.

(iii) *R preserves decomposability if and only if there exists a well-defined linear map* $\Lambda^2R\colon \Lambda^4V \to \Lambda^4V$ *satisfying the equation* $\Lambda^2R(\alpha\beta) = R\alpha R\beta$ *for all* $\alpha, \beta \in \Lambda^2V$.

(iv) *If R preserves decomposability and is symmetric, then for all* $\alpha, \beta \in \operatorname{im} R$, $\alpha\beta = 0$ *if and only if* $R\alpha R\beta = 0$.

DEFINITION 3.2. Let $U \subset V$ be the subspace spanned by all planes $\{\omega\}$ belonging to decomposable bivectors $\omega \in \operatorname{im} R$.

PROPOSITION 3.3. (i) *If* im R *is spanned by decomposable bivectors* $\omega_1, \ldots, \omega_N$, *then U is spanned by the planes* $\{\omega_1\}, \ldots, \{\omega_N\}$ *of V, and* $\operatorname{im} R \subset \Lambda^2U$.

(ii) *If R preserves decomposability, then* $\operatorname{im} R \subset \Lambda^2U$.

Definition 3.2 is suitable for defining U only when R preserves decomposability. We shall indicate some more general definitions for U, since these permit dim U to be related to other known numerical invariants of R. Define the absolute nullspace of a curvature operator R to be the subspace N in V consisting of all vectors x such that $Rxy = 0$ for all y in V; define its "rank" subspace to be the smallest subspace W in V such that $\operatorname{im} R \subset \Lambda^2W$. Then dim N is called the nullity index of R (cf. [**5**, p. 347]), and dim W is called the "rank" of R (cf. [**11**, §9 on p. 34 and §4 on p. 76]). The codimension of N is called the conullity or *type* of R. Clearly, it can also be described as the rank of the linear mapping $F\colon V \to \operatorname{Hom}(V, \Lambda^2V)$, where $F(x)y = Rxy$ (cf. [**8**, p. 186]).

PROPOSITION 3.4. (i) $W = N^\perp$; *hence* "rank" R = type R.

(ii) *If R preserves decomposability, then* $U = W = N^\perp$; *hence* dim U = type R.

(iii) *If* $R = \pm\Lambda^2L$ *for some linear map* $L\colon V \to V$, *then* $U = \operatorname{im} L$; *hence* dim U = type R = rank L.

4. Factoring im R when $\Lambda^2R \not\equiv 0$.

Henceforth we assume R is a decomposability-preserving curvature operator. In this section we obtain the factorization $\operatorname{im} R = \Lambda^2U$ for the case $\Lambda^2R \not\equiv 0$.

Any pair of decomposable bivectors α_1, α_2 with $\alpha_1\alpha_2 \neq 0$ can be written as $\alpha_1 = a_1a_2$, $\alpha_2 = a_3a_4$, where a_1, a_2, a_3, a_4 are independent. If we suppose that $R\alpha_1R\alpha_2 \neq 0$, then we can similarly write $R\alpha_1 = b_1b_2$, $R\alpha_2 = b_3b_4$, where b_1, b_2, b_3, b_4 are independent.

LEMMA 4.1. *We may assume that*

$$Ra_1a_2 = b_1b_2, \qquad Ra_1a_3 = b_1b_3,$$

$$\left.\begin{matrix} Ra_2a_3 \\ Ra_1a_4 \end{matrix}\right\} = \pm\left\{\begin{matrix} b_2b_3 \\ b_1b_4 \end{matrix}\right. \quad \text{or} \quad \pm\left\{\begin{matrix} b_1b_4 \\ b_2b_3 \end{matrix}\right. \qquad (\textit{both } + \textit{ or } -),$$

$$Ra_2a_4 = b_2b_4, \qquad Ra_3a_4 = b_3b_4.$$

PROOF. Consider the planes $\{Ra_ia_j\}$ in U. Define vectors b_1, b_2, b_3, b_4 by the conditions

$$\{Ra_1a_2\} \cap \{Ra_1a_3\} = \{b_1\}, \qquad \{Ra_1a_2\} \cap \{Ra_2a_4\} = \{b_2\},$$
$$\{Ra_1a_3\} \cap \{Ra_3a_4\} = \{b_3\}, \qquad \{Ra_2a_4\} \cap \{Ra_3a_4\} = \{b_4\}.$$

Then these b_i are independent and $Ra_1a_2 = x_1b_1b_2$, $Ra_3a_4 = x_2b_3b_4$, $Ra_1a_3 = x_3b_1b_3$, $Ra_2a_4 = x_4b_2b_4$, where x_i are nonzero scalars. Put $b_1' = x_3b_1$, $b_2' = x_1b_2/x_3$, $b_3' = b_3$, $b_4' = x_2b_4$; then $Ra_1a_2 = b_1'b_2'$, $Ra_3a_4 = b_3'b_4'$, and $Ra_2a_4 = yb_2'b_4'$. But now Proposition 3.1(ii) implies that $b_1'b_2'b_3'b_4' = -yb_1'b_3'b_2'b_4'$, whence $y = 1$. Dropping the primes, we have the first two and the last two of the desired equations.

To get the remaining equations, set $\alpha = Ra_1a_4$, $\beta = Ra_2a_3$. Then Proposition 3.1(ii) implies $\alpha\beta = b_1b_2b_3b_4$, which in turn implies $\alpha, \beta \in \Lambda^2\{b_1, b_2, b_3, b_4\}$. Moreover, since α, β are both adjacent to the four bivectors b_1b_2, b_1b_3, b_2b_4 and b_3b_4, we must have

$$\alpha = xb_2b_3 + yb_1b_4 \quad \text{and} \quad \beta = zb_2b_3 + wb_1b_4,$$

where x, y, z and w are scalars. Since $\alpha\alpha = 0$ and $\beta\beta = 0$, and $\alpha\beta = b_1b_2b_3b_4$, it follows that $xy = 0$, $zw = 0$ and $xw + yz = 1$. This implies that either $x = 0$, $w = 0, yz = 1$, or $y = 0$, $z = 0$, $xw = 1$. Therefore, these two cases hold.

Case (i). $\alpha = zb_1b_4$, $\beta = z^{-1}b_2b_3$ or

Case (ii). $\alpha = zb_2b_3$, $\beta = z^{-1}b_1b_4$.

For Case (i), set $b_1' = |z|^{1/2}b_1$, $b_2' = |z|^{-1/2}b_2$, $b_3' = |z|^{-1/2}b_3$, $b_4' = |z|^{1/2}b_4$. Then $b_1'b_2' = b_1b_2$, $b_1'b_3' = b_1b_3$, $b_2'b_4' = b_2b_4$, $b_3'b_4' = b_3b_4$. However, $b_2'b_3' = |z|^{-1}b_2b_3 = \pm z^{-1}b_2b_3 = \pm\beta = \pm Ra_2a_3$, and $b_1'b_4' = |z|b_1b_4 = \pm\alpha = \pm Ra_1a_4$, where both signs are $+$ when $z > 0$ and both are $-$ when $z < 0$.

For Case (ii) set $b_1' = |z|^{-1/2}b_1$, $b_2' = |z|^{1/2}b_2$, $b_3' = |z|^{1/2}b_3$, $b_4' = |z|^{-1/2}b_4$. Then $b_1'b_2' = b_1b_2$, $b_1'b_3' = b_1b_3$, $b_2'b_4' = b_2b_4$ and $b_3'b_4' = b_3b_4$. However, $b_2'b_3' = |z|b_2b_3 = \pm zb_2b_3 = \pm\alpha = \pm Ra_1a_4$, and $b_1'b_4' = |z|^{-1}b_1b_4 = \pm z^{-1}b_1b_4 = \pm\beta = \pm Ra_2a_3$, with the same type of sign. Q.E.D.

LEMMA 4.2. *Let ω_1 and ω_2 be decomposable bivectors in* im R, *such that $\omega_1\omega_2 \neq 0$. Then the* 4-*dimensional subspace $U_1 = \{\omega_1\} + \{\omega_2\}$ of V has the property that* $\Lambda^2U_1 \subset \operatorname{im} R$.

PROOF. Let $\omega_1 = R\alpha_1$ and $\omega_2 = R\alpha_2$. By the orthogonal decomposition of Λ^2V given in §2, it may be assumed that α_1 and α_2 are in im R. But then Proposition 3.1(iv) implies that both α_1 and α_2 are decomposable and that $\alpha_1\alpha_2 \neq 0$. Hence α_1, α_2 can be expressed as $\alpha_1 = a_1a_2$, $\alpha_2 = a_3a_4$. Let $\omega_1 = b_1b_2$ and $\omega_2 = b_3b_4$. Then $\{b_1, b_2, b_3, b_4\} + \{b_1b_2\} = \{b_3b_4\} = \{\omega_1\} + \{\omega_2\} = U_1$. Now Lemma 4.1 implies that each of the bivectors $b_1b_2, \ldots, b_3b_4$ in the basis of Λ^2U_1 is an image by R of a bivector in Λ^2V; hence Λ^2U_1 must be contained in im R. Q.E.D.

PROPOSITION 4.3. *If R: $\Lambda^2V \to \Lambda^2V$ is a curvature operator which preserves decomposability, and if $\Lambda^2R \not\equiv 0$, then* im $R = \Lambda^2U$. *Moreover,* dim $U \geqslant 4$ *and* rank $R \geqslant 6$.

PROOF. Let U be defined by Definition 3.2. Then by Proposition 3.3(ii), im $R \subset \Lambda^2 U$, and it suffices to prove that $\Lambda^2 R \not\equiv 0$ implies $\Lambda^2 U \subset$ im R. To establish this inclusion, it is enough to show that each decomposable bivector of $\Lambda^2 U$ is in im R, i.e., that $x, y \in U$ implies $xy \in$ im R.

Let $x, y \in U$. By definition of U, there exist decomposable bivectors $\omega_1, \omega_2 \in$ im R such that $x \in \{\omega_1\}, y \in \{\omega_2\}$.

If $\omega_1\omega_2 \neq 0$, then Lemma 4.2 implies that $\Lambda^2 U_1 \subset$ im R, where $U_1 = \{\omega_1\} + \{\omega_2\}$. But both x and y are in U_1, so that xy is in $\Lambda^2 U_1$; hence $xy \in$ im R.

If $\omega_1\omega_2 = 0$, then we have $\omega_1 = xz$, $\omega_2 = zy$ for a suitable vector z, and we may assume x, y, z independent. In this case Lemma 4.2 cannot be applied directly, so an auxiliary subspace must be used from which to get decomposable vectors of im R that are not adjacent to ω_1 or ω_2. For this purpose, let $e_1, \ldots, e_n$ be a basis of V. Then Proposition 3.1(iii) implies that if $\Lambda^2 R \not\equiv 0$ then not all $Re_ie_jRe_ke_l$ with $i < j < k < l$ can vanish. By relabeling the basis, it may be assumed that $Re_1e_2Re_3e_4 \neq 0$. Then Lemma 4.2 implies that $\Lambda^2 U_0 \subset$ im R, where $U_0 = \{Re_1e_2\} + \{Re_3e_4\}$. Let us consider the relative positions of the 4-dimensional space U_0 and the 3-dimensional space $W_0 = \{\omega_1\} + \{\omega_2\} = \{x, y, z\}$.

If $W_0 \subset U_0$, then $x, y \in U_0$, so $xy \in \Lambda^2 U_0$, whence $xy \in$ im R, and we are done. On the other hand, if $W_0 \not\subset U_0$, then $U_0 \cap W_0$ has dimension $\leqslant 2$. Hence $(U_0 \cap W_0)^\perp \cap U_0$ has dimension $\geqslant 2$ and therefore contains a pair of independent vectors v_1, v_2. Setting $\omega_3 = v_1v_2$, we have $\omega_3 \in$ im R, $\omega_2\omega_3 \neq 0$ and z, y, v_1, v_2 independent. Now we can apply Lemma 4.2 to ω_2, ω_3 to conclude $\Lambda^2 U_1 \subset$ im R, where $U_1 = \{\omega_2\} + \{\omega_3\}$.

If $\{\omega_1\} \subset U_1$, then $x, y \in U_1$, whence $xy \in$ im R, and we are done. If $\{\omega_1\} \not\subset U_1$, put $\omega_4 = yv_2$; then $\omega_1\omega_4 \neq 0$, because $\omega_1\omega_4 = 0$ would imply $\{\omega_1\} \subset U_1$. We again apply Lemma 4.2 to ω_1, ω_4, to conclude $\Lambda^2 U_2 \subset$ im R, where $U_2 = \{\omega_1\} + \{\omega_4\} = \{x, y, z, v_2\}$. Thus $xy \in$ im R. Q.E.D.

5. Correlations. The special case $\Lambda^2 R \not\equiv 0$, rank $R = 6$ presents an anomalous feature–the possibility of R being a correlation–which must be excluded later. Therefore we include here a discussion of this case.

It follows from Proposition 4.3 that im $R = \Lambda^2 U$ and dim $U = 4$. Hence the nonsingular map $R|$im R: $\Lambda^2 U \to \Lambda^2 U$ satisfies the equations of Lemma 4.1, where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are now two bases of U. If $Ra_2a_3 = \pm b_2b_3$ and $Ra_1a_4 = \pm b_1b_4$, R is called a *collineation*, whereas if $Ra_2a_3 = \pm b_1b_4$ and $Ra_1a_4 = \pm b_2b_3$, R is a *correlation*.

If our R has the form $R = \pm \Lambda^2 L$, then clearly it is a collineation. Hence for the case $\Lambda^2 R \not\equiv 0$, rank $R = 6$, the condition that R is not a correlation is necessary for R to admit factorization as $R = \pm \Lambda^2 L$.

The following three results give criteria for deciding whether R is a collineation or a correlation.

PROPOSITION 5.1. *Let R: $\Lambda^2 V \to \Lambda^2 V$ be a decomposability-preserving curvature operator having rank* 6. *Let $e_1, \ldots, e_n$ be a basis of V such that $e_1, \ldots, e_4$ is a basis of U. If $Re_1e_2 = b_1b_2$, $Re_1e_3 = b_1b_3$, $Re_2e_3 = \pm b_2b_3$, then R is a collineation,* *whereas if* $Re_2e_3 = \pm b_1b_4$, *then* R *is a correlation.* (*The vectors* $b_1, \ldots, b_4$ *are a suitable independent set in* U.)

PROOF. Clear.

PROPOSITION 5.2. *R is a collineation or a correlation according to whether* $\{Ra_1a_2\} + \{Ra_1a_3\} + \{Ra_2a_3\}$ *is a* 3- *or* 4-*dimensional subspace of* V, *respectively, where* a_1, a_2, a_3 *is an arbitrary independent set in* V.

PROOF. Consider the mapping of planes of V into planes of V defined by $\{xy\} \to \{Rxy\}$. If R is a collineation, then the set of planes containing a given line gets mapped onto a set of planes that also contain a common line, and the set of planes contained in a given 3-space gets mapped onto a set of the same type. But if R is a correlation, these types of sets of planes get reversed (cf. [3, p. 107]). Q.E.D.

PROPOSITION 5.3. *Let* R: $\Lambda^2V \to \Lambda^2V$ *be a decomposability-preserving curvature operator having rank* 6. *Let* $e_1, \ldots, e_n$ *be a basis of* V *such that* e_1, e_2, e_3, e_4 *is a basis of* U, *and let* R^{ij}_{kl} *be the matrix of* R *with respect to the basis* e_ie_j, $i < j$, *of* Λ^2V. *Then* R *is a collineation if and only if the following matrix has rank* 4.

$$\begin{bmatrix} R^{23}_{12} & -R^{13}_{12} & R^{12}_{12} & 0 \\ R^{24}_{12} & -R^{14}_{12} & 0 & R^{12}_{12} \\ R^{23}_{13} & -R^{13}_{13} & R^{12}_{13} & 0 \\ R^{24}_{13} & -R^{14}_{13} & 0 & R^{12}_{13} \\ R^{23}_{23} & -R^{13}_{23} & R^{12}_{23} & 0 \\ R^{24}_{23} & -R^{14}_{23} & 0 & R^{12}_{23} \end{bmatrix}$$

PROOF. For a decomposable bivector α in Λ^2U the plane $\{\alpha\}$ spanned by it in V is given by

$$\{\alpha\} = \{x|\alpha x = 0\} = \{x|x^i\alpha^{jk} - x^j\alpha^{ik} + x^k\alpha^{ij} = 0 \text{ for } 1 \leqslant i < j < k \leqslant 4\},$$

where $\alpha = \Sigma\alpha^{ij}e_ie_j$ with $1 \leqslant i < j \leqslant 4$. Since α is decomposable, $\alpha\alpha = 0$, which gives $\alpha^{12}\alpha^{34} - \alpha^{13}\alpha^{24} + \alpha^{14}\alpha^{23} = 0$. From this equation it can be seen that the four equations describing $\{\alpha\}$ are dependent, and that those two with indices i, j, k equal to 1, 2, 3 and 1, 2, 4 are independent. Hence, as a subset of $\mathbf{R}^4$, $\{\alpha\}$ is the nullspace of the matrix

$$\begin{pmatrix} \alpha^{23} & -\alpha^{13} & \alpha^{12} & 0 \\ \alpha^{24} & -\alpha^{14} & 0 & \alpha^{12} \end{pmatrix}.$$

If $\alpha = Re_1e_2$, then the corresponding matrix is given by the first two rows of the 6×4 matrix in the statement of this proposition; its nullspace is precisely $\{Re_1e_2\}$, considered as a subspace of $\mathbf{R}^4$. The next two pairs of rows of that matrix correspond to $\{Re_1e_3\}$ and $\{Re_2e_3\}$. Therefore, the nullspace of the whole 6×4 matrix corresponds to the space $\{Re_1e_2\} \cap \{Re_1e_3\} \cap \{Re_2e_3\}$. Now R is a collineation if and only if this space is $\{0\}$, which occurs exactly when the 6×4 matrix has maximal rank. Q.E.D.

6. The case $\Lambda^2 R \equiv 0$. It is easy to see from Proposition 3.1 that $\Lambda^2 R \equiv 0$ if and only if $R(x_1 x_2)R(x_3 x_4) = 0$ for all x_1, x_2, x_3, x_4 in V. This is a rather severe restriction on R; as a consequence, we can perform the factorings $\operatorname{im} R = \Lambda^2 U$ and $R = \pm\Lambda^2 L$ both at the same time.

PROPOSITION 6.1. *Let* R: $\Lambda^2 V \to \Lambda^2 V$ *be a curvature operator which preserves decomposability and satisfies* $\Lambda^2 R \equiv 0$. *Then* $R = \pm\Lambda^2 L$ *for a suitable linear map* L: $V \to V$ *if and only if* rank $R = 0$ *or* 1, *or* rank $R =$ type $R = 3$. *Moreover, in case* rank $R = 0$ *or* 1, *the* + *sign may be assumed to hold, with* L *symmetric.*

PROOF. Let $a_1, \ldots, a_k$ be a basis of U, with U as in Definition 3.2. Since R preserves decomposability, Proposition 3.3 states that $\operatorname{im} R \subset \Lambda^2 U$. Hence R restricts to a linear map $\Lambda^2 U \to \Lambda^2 U$ which still has the same set as image. Therefore the bivectors $Ra_i a_j$ for $1 \leqslant i < j \leqslant k$ span $\operatorname{im} R$, and $U = \Sigma\{Ra_i a_j\}$ accordingly. Since $\Lambda^2 R \equiv 0$, all the nonzero bivectors among these $Ra_i a_j$ must be adjacent to each other, so that all the corresponding planes $\{Ra_i a_j\}$ have 1-dimensional intersections. But then the result of [**1**, p. 16] implies that either (i) all of them contain one 1-dimensional subspace of V or (ii) they are all contained in one 3-dimensional subspace of V.

In case (i), $Ra_i a_j = b_1 b_{ij}$ for all $1 \leqslant i < j \leqslant k$, where b_1 is a fixed vector in U, and b_{ij} is taken equal to b_1 when $Ra_i a_j = 0$. By definition, U is spanned by the planes $\{Ra_i a_j\} = \{b_1, b_{ij}\}$ corresponding to the nonzero $Ra_i a_j$. This means that these vectors $b_1, \ldots, b_{ij}, \ldots$ span U, which implies that k of them are independent, including b_1. Denote these vectors by $b_1, b_2, \ldots, b_k$; they are a basis of U, and consequently the $k - 1$ bivectors $b_1 b_2, \ldots, b_1 b_k$ are a basis of $\operatorname{im} R$. Therefore R has rank $k - 1$.

In case (ii), all the planes $\{Ra_i a_j\}$ lie in one 3-dimensional subspace of V, whence U has dimension $k \leqslant 3$.

Now suppose $R = \pm\Lambda^2 L$ for some linear map L: $V \to V$. It follows from Proposition 3.4 that $U = \operatorname{im} L$ and $\Lambda^2 U = \operatorname{im} R$. This implies that rank $R = \dim \Lambda^2 U = \binom{k}{2}$. In Case (i), this means $\binom{k}{2} = k - 1$, which implies $k = 1$ or 2, so that rank $R = 0$ or 1. In Case (ii), $k \leqslant 3$, i.e., $k = 0, 1, 2$ or 3. Therefore, rank $R = \binom{k}{2} = 0$, 1, or 3. Hence $R = \pm\Lambda^2 L$ implies the rank and dimension conditions in the proposition, noting that by Proposition 3.4(ii), $\dim U =$ type R.

To prove the converse, start with rank $R = 0$. Then $R \equiv 0$, so $R = \Lambda^2 L$ for $L \equiv 0$. Next, suppose rank $R = 1$. This implies that $Re_i e_j \neq 0$ for some $i < j$, where $e_1, \ldots, e_n$ is any basis of V. This nonzero $Re_i e_j$ must be decomposable, since R preserves decomposability; it also must span $\operatorname{im} R$. Let v_1, v_2 be an orthonormal basis of the corresponding plane $\{Re_i e_j\}$. Therefore $v_1 v_2$ also spans $\operatorname{im} R$; in fact, $\operatorname{im} R = \Lambda^2 U$, where $U = \{v_1, v_2\}$. Since $R|\operatorname{im} R$ is nonsingular, there is a scalar $c \neq 0$ such that $Rv_1 v_2 = cv_1 v_2$. Extend v_1, v_2 to an orthonormal basis $v_1, v_2, v_3, \ldots, v_n$ of V; then $v_1 v_2, v_1 v_3, \ldots, v_{n-1} v_n$ is an orthonormal basis of $\Lambda^2 V$. Since R is symmetric, $\Lambda^2 V$ splits into the orthogonal sum $\Lambda^2 V = \ker R + \operatorname{im} R$. This implies that $v_1 v_3, \ldots, v_{n-1} v_n$ is a basis of $\ker R$, so that $Rv_i v_j = 0$ when

$(i,j) \neq (1,2)$. Now set $Lv_1 = cv_1$, $Lv_2 = v_2$, $Lv_i = 0$ for $3 \leqslant i$. Then $R = \Lambda^2 L$. Note that L is symmetric, and that rank $R = 1$ implies $\Lambda^2 R \equiv 0$.

Finally, suppose that rank R = type $R = 3$, i.e., $\dim U = 3$. By the first part of this proof, three of the bivectors Ra_ia_j for $1 \leqslant i < j \leqslant k$, are independent and their planes span U. Now either Case (i) or Case (ii) holds.

In Case (i), it was shown above that U has a basis $b_1, \dots, b_k$ and that the bivectors $b_1b_2, \dots, b_1b_k$ are a basis of im R. Since rank $R = 3$, this basis contains exactly 3 bivectors, which implies $k = 4$. But this contradicts the hypothesis that $\dim U = 3$, which rules out this case.

In Case (ii), the planes $\{Ra_ia_j\}$ are all contained in a 3-dimensional subspace of V, which implies $\dim U \leqslant 3$, so that also $\dim \Lambda^2 U \leqslant 3$. Since by Proposition 3.3 im $R \subset \Lambda^2 U$ and, by hypothesis, rank $R = 3$, we have im $R = \Lambda^3 U$ and $\dim U = 3$. Let v_1, v_2, v_3 be an orthonormal basis of U. Then v_1v_2, v_1v_3, v_2v_3 is an orthonormal basis of $\Lambda^2 U$ = im R and the planes $\{Rv_1v_2\}$, $\{Rv_1v_3\}$, $\{Rv_2v_3\}$ span U.

Consequently, these planes intersect each other in three distinct lines $\{b_1\}$, $\{b_2\}$, $\{b_3\}$, where the vectors b_1, b_2, b_3 can be chosen so that $Rv_1v_2 = b_1b_2$, $Rv_1v_3 = b_1b_3$, $Rv_2v_3 = cb_2b_3$, with c a nonzero scalar. It is not hard to see that the lengths and directions of the vectors b_1, b_2, b_3 can be adjusted to get $Rv_1v_2 = b_1b_2$, $Rv_1v_3 = b_1b_3$, $Rv_2v_3 = \pm b_2b_3$. Now extend the basis v_1, v_2, v_3 to an orthonormal basis $v_1, v_2, v_3, v_4, \dots, v_n$ of V. Then v_1v_2, v_1v_3 $v_1v_4, \dots, v_{n-1}v_n$ is an orthonormal basis of $\Lambda^2 V$. Just as in the case of rank $R = 1$ above, it follows from the symmetry of R that $Rv_iv_j = 0$ when (i,j) is not one of the pairs (1, 2), (1, 3), (2, 3).

Set $Lv_1 = \pm b_1$, $Lv_2 = b_2$, $Lv_3 = b_3$ and $Lv_i = 0$ for $4 \leqslant i$. Then $R = \pm\Lambda^2 L$. Q.E.D.

7. The main factorization theorems. Here we bring together the results of the previous sections to obtain first $R = \pm\Lambda^2 L$ and then $R = \Lambda^2 L$. As indicated in §2, we shall use the results of **[9]** in a formulation extended to include the case $\dim V = 4$.

PROPOSITION 7.1. *A nonsingular curvature operator* $R\colon \Lambda^2 V \to \Lambda^2 V$, *with* $\dim V \geqslant 4$, *can be factored as* $R = \pm\Lambda^2 L$, *for a suitable linear map* $L\colon V \to V$, *if and only if* R *preserves decomposability and is not a correlation in the case* $\dim V = 4$.

PROOF. This is just [**9**, Theorem 1], extended to include $\dim V = 4$ (cf. [**2**, p. 38]). By Proposition 5.1, maps of form $R = \pm\Lambda^2 L$ are collineations, not correlations. Q.E.D.

Define two scalar invariants of R as follows (cf. [**9**, p. 201]). Set

$$\phi(R) = R^{ij}_{kl} R^{kp}_{iq} R^{lq}_{jp}, \qquad \psi(R) = R^{ij}_{kl} R^{kl}_{pq} R^{pq}_{ij},$$

where summation is understood over repeated indices, and R^{ij}_{kl} is the matrix of R with respect to any basis e_ie_j, $i < j$, corresponding to a basis $e_1, \dots, e_n$ of V.

PROPOSITION 7.2 [9, THEOREM 3]. *Suppose R is a nonsingular curvature operator such that $R = \pm\Lambda^2 L$ for a symmetric linear map L: $V \to V$, with* $\dim V \geqslant 3$. *Then $R = \Lambda^2 L$ if and only if $\phi(R) + \frac{1}{4}\psi(R) > 0$. Moreover, in case $n \equiv 3 \pmod 4$, this inequality may be replaced by* $\det R > 0$.

THEOREM 7.3. *Let R: $\Lambda^2 V \to \Lambda^2 V$ be a curvature operator. Then $R = \pm\Lambda^2 L$ for some linear map L: $V \to V$ if and only if R preserves decomposability and satisfies one of the following additional conditions.*

(i) $\Lambda^2 R \not\equiv 0$ *and R is not a correlation in case* rank $R = 6$.

(ii) $\Lambda^2 R \equiv 0$ *and either* rank $R = 0$ *or* 1, *or* rank $R =$ type $R = 3$.

PROOF. Suppose $\Lambda^2 R \not\equiv 0$. If R preserves decomposability, then by Proposition 4.3, rank $R \geqslant 6$ and im $R = \Lambda^2 U$, with $\dim U \geqslant 4$. If R is nonsingular, then $U = V$. If R is not a correlation, then $R = \pm\Lambda^2 L$ follows from Proposition 7.1. If R is singular, then by §2, $R_1 = R|\text{im } R$: $\Lambda^2 U \to \Lambda^2 U$ is a nonsingular symmetric linear map. By Proposition 7.1, it factors as $R_1 = \pm\Lambda^2 L_1$, provided R_1 is not a correlation (note that R is a correlation if and only if R_1 is one). Therefore §2 gives $R = \pm\Lambda^2 L$, for a linear L: $V \to V$. Thus if R preserves decomposability and satisfies (i), then $R = \pm\Lambda^2 L$.

Suppose now that $\Lambda^2 R \equiv 0$ and let R preserve decomposability. Then Proposition 6.1 shows that (ii) implies $R = \pm\Lambda^2 L$.

Conversely, assume $R = \pm\Lambda^2 L$; clearly R preserves decomposability. If $\Lambda^2 R \equiv 0$, then Proposition 6.1 implies statement (ii). If $\Lambda^2 R \not\equiv 0$ and rank $R = 6$, then Proposition 5.1 implies that R is not a correlation. Q.E.D.

COROLLARY 7.4. *If R: $\Lambda^2 V \to \Lambda^2 V$ is a curvature operator that preserves decomposability and has rank > 6, then it admits factorization as $R = \pm\Lambda^2 L$ if and only if $\Lambda^2 R \not\equiv 0$.*

In order to state the next theorem, we recall the Bianchi identity of Riemannian geometry. R is said to satisfy the *Bianchi identity* if

$$\langle Rx_1x_2, x_3x_4\rangle + \langle Rx_2x_3, x_1x_4\rangle + \langle Rx_3x_1, x_2x_4\rangle = 0$$

for all x_1, x_2, x_3, x_4 in V.

THEOREM 7.5. *Suppose R is a curvature operator such that $R = \pm\Lambda^2 L$ for a linear map L: $V \to V$ with rank $L \geqslant 3$. Then L is symmetric if and only if R satisfies the Bianchi identity.*

PROOF. Substituting $R = \pm\Lambda^2 L$ into the left side of the Bianchi identity gives

$$\langle Lx_3, x_4\rangle(\langle Lx_2, x_1\rangle - \langle Lx_1, x_2\rangle) + \langle Lx_2, x_4\rangle(\langle Lx_1, x_3\rangle - \langle Lx_3, x_1\rangle) + \langle Lx_1, x_4\rangle(\langle Lx_3, x_2\rangle - \langle Lx_2, x_3\rangle).$$

If L is symmetric, this clearly reduces to zero. Conversely, if the Bianchi identity holds, this expression is identically zero. Consider any two vectors x_1, x_2. It was observed in the proof of Proposition 6.1 that $R = \pm\Lambda^2 L$ implies $U = \text{im } L$ and $\Lambda^2 U = \text{im } R$. Hence, rank $R = \binom{k}{2}$, where $k = \dim U$. Consequently, the hypothesis, rank $L \geqslant 3$, implies that $\dim U \geqslant 3$. Therefore the vectors Lx_1, Lx_2 are in U

and do not span all of U. Hence there exists a nonzero vector $x_4 \in U$ such that $x_4 \perp$ both Lx_1 and Lx_2. This gives the equation

$$\langle Lx_3, x_4\rangle(\langle Lx_2, x_1\rangle - \langle Lx_1, x_2\rangle) = 0.$$

Now the map $L_1^{-1} = (L|U)^{-1}\colon U \to U$ is well defined. Set $x_3 = L_1^{-1}x_4$; then $Lx_3 = L_1x_3 = x_4$ and consequently $\langle Lx_3, x_4\rangle = \langle x_4, x_4\rangle \neq 0$, since x_4 was chosen nonzero. Therefore, the term in parentheses in the above equation vanishes, whence L is symmetric. Q.E.D.

THEOREM 7.6. *Suppose R is a curvature operator such that $R = \pm\Lambda^2 L$ for a symmetric linear map $L\colon V \to V$ having rank $L \geqslant 3$. Then $R = \Lambda^2 L$ if and only if $\phi(R) + \frac{1}{4}\psi(R) > 0$. Moreover, in case $n \equiv 3$ (mod 4), this inequality may be replaced by* $\det(R|\text{im } R) > 0$.

PROOF. Since $L\colon V \to V$ is a symmetric linear map, V splits into the orthogonal sum $V = \ker L + \text{im } L$, and $U = \text{im } L$. Then each $x \in V$ can be written as $x = x_0 + x_1$ with $x_0 \in \ker L$ and $x_1 \in U$. Note $Lx = Lx_1 = L_1x_1$ where $L_1 = L|U\colon U \to U$ is a nonsingular symmetric linear map. This implies

$$Rxy = \pm LxLy = \pm L_1x_1L_1y_1 = \pm\Lambda^2L_1(x_1y_1).$$

As noted in §2 the space Λ^2V can be written as $\Lambda^2V = \ker R + \text{im } R$, with $\Lambda^2U = \text{im } R$ here. Then $xy = (x_0 + x_1)(y_0 + y_1) = x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1 = (xy)_0 + (xy)_1$ and $(xy)_1 = x_1y_1$. Hence $Rxy = R(xy)_1 = R_1x_1y_1$, where $R_1 = R|\Lambda^2U$ is a nonsingular symmetric linear map.

It follows from the above two equations for Rxy that $R = \Lambda^2L$ if and only if $R_1 = \Lambda^2L_1$.

Let $e_1, \ldots, e_n$ be a basis of V such that $e_1, \ldots, e_k$ is a basis of $U = \text{im } L$ and $e_{k+1}, \ldots, e_n$ is a basis of $\ker L$. Then $Re_je_j = R_1(e_ie_j)$ for $1 \leqslant i, j \leqslant k$ and $Re_ie_p = Re_pe_q = 0$ for $k + 1 \leqslant p, q \leqslant n$. It follows that $\phi(R) = \phi(R_1)$ and $\psi(R) = \psi(R_1)$, since all terms not involving only the matrix of R_1 must vanish. Therefore $\phi(R) + \frac{1}{4}\psi(R) = \phi(R_1) + \frac{1}{4}\psi(R_1)$, and the conclusion of the theorem follows from Proposition 7.2 applied to $R_1\colon \Lambda^2U \to \Lambda^2U$ and $L_1\colon U \to U$. Q.E.D.

REMARK. The sign in $\pm\Lambda^2L$ actually determines two disjoint classes of maps. Namely, if $L, M\colon V \to V$ are symmetric linear maps of rank $\geqslant 3$, then $\Lambda^2L \neq -\Lambda^2M$. This follows from the remark in [9, top of p. 201], via the equation $\Lambda^2L(xy) = \Lambda^2L_1(x_1y_1)$ obtained in the proof of Theorem 7.6 above.

The following theorem is the main result of this paper.

THEOREM 7.7. *A curvature operator $R\colon \Lambda^2V \to \Lambda^2V$ of rank > 1 can be factored as $R = \Lambda^2L$, for a suitable symmetric linear map $L\colon V \to V$, if and only if it satisfies each of the following conditions.*

(i) *R preserves decomposability.*

(ii) *R satisfies the Bianchi identity.*

(iii) $\phi(R) + \frac{1}{4}\psi(R) > 0$. (*If* $n \equiv 3$ (mod 4), *then* $\det(R|\text{im } R) > 0$ *may be substituted.*)

(iv) *Either* $\Lambda^2R \not\equiv 0$ *and R is not a correlation if* rank $R = 6$ *or* $\Lambda^2R \equiv 0$ *and* rank $R =$ type $R = 3$.

In case rank $R \leqslant 1$, *the conditions* (ii), (iii), *and* (iv) *are deleted.*

PROOF. For rank $R > 1$, see Theorems 7.3, 7.5, 7.6 and the subsequent Remark. For rank $\leqslant 1$, see Proposition 6.1 and its proof; in this case $\Lambda^2 R \equiv 0$. Q.E.D.

In conclusion, we make some observations about curvature operators not factorizable as $R = \Lambda^2 L$. First, if one is willing to concede condition (iii) above, then one should consider the factorization $R = \pm\Lambda^2 L$. Second, if one wants R to preserve decomposability and have rank > 3, then Theorem 7.3 and Corollary 7.4 force $\Lambda^2 R \equiv 0$.

EXAMPLE. Let $e_1, \ldots, e_n$ be an othonormal basis of V and define (for $i < j$)

$$Re_ie_j = \begin{cases} e_1e_j, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\Lambda^2 R \equiv 0$ and R preserves decomposability. Since rank $R = n - 1$, this R is not factorizable as $R = \pm\Lambda^2 L$ when $n > 4$. Note that $U = V$ and that R satisfies the Bianchi identity.

The general nonfactorizable curvature operator does not preserve decomposability. If it satisfies the Bianchi identity, then it has the decomposition $R = \Lambda^2 L_1 + \cdots + \Lambda^2 L_p$ mentioned in §1 (cf. [**4**, p. 102]).

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